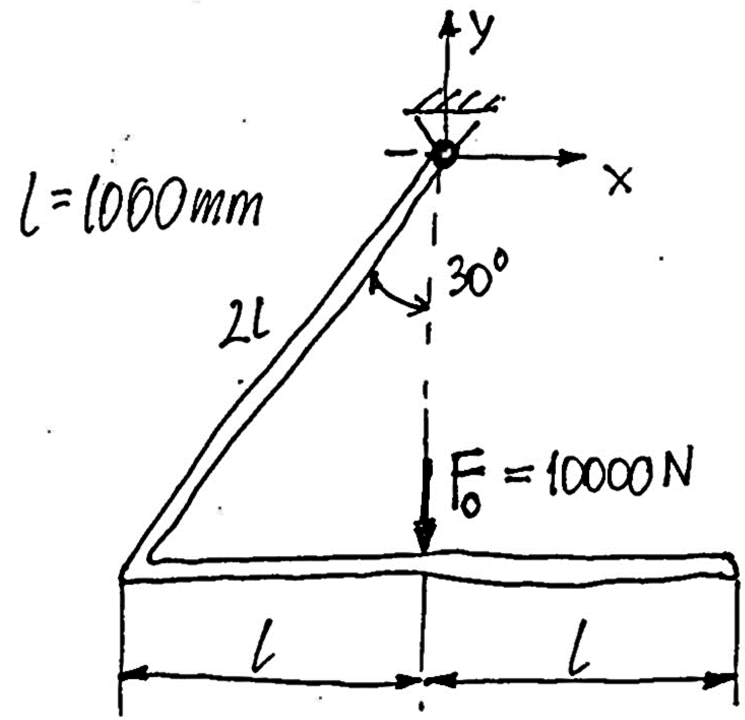
Politechnika Warszawska | Wydział Mechaniczny Energetyki i Lotnictwa | Zakład Wytrzymałości Materiałów i Konstrukcji

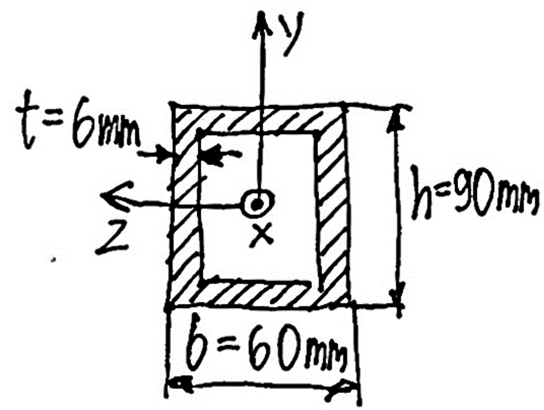
Metoda elementów skończonych (MES1)

Wykład 10C. Rama płaska. Przykład 1.

05.2022

Przykład Zbuduj model MES ramy 2D. Wyznacz przemieszczenia węzłowe, naprężenia, siły wewnętrzne i reakcje. Sprawdź warunki równowagi.





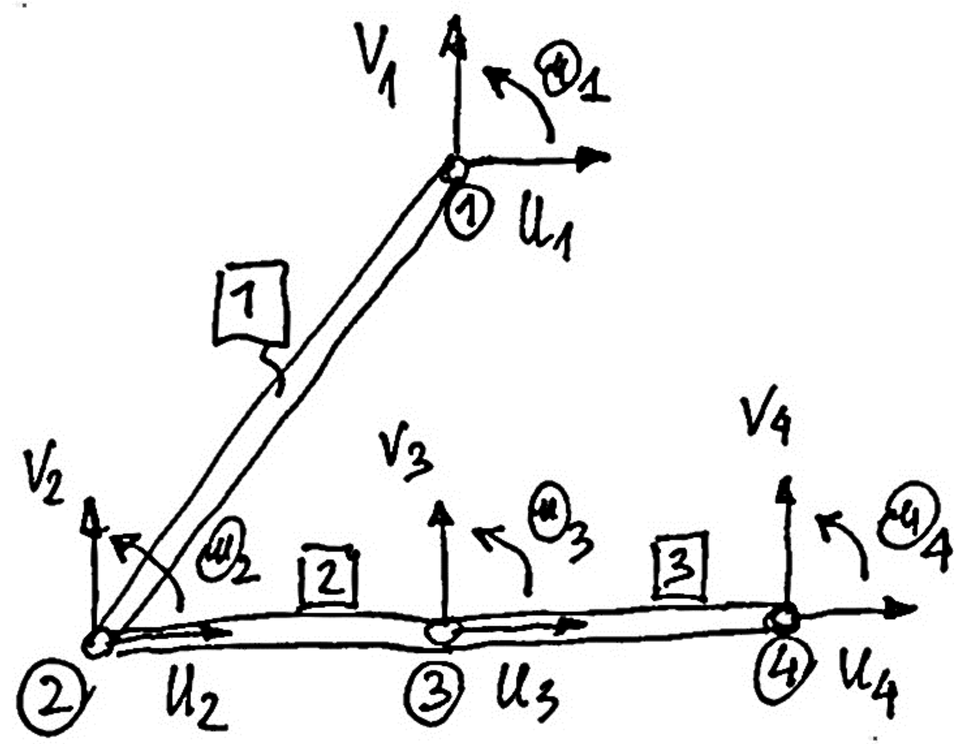
$$A = b \cdot h - (b-2t)(h-2t) = 1656 \ mm^2$$

$$J_z = \frac{bh^3}{12} - \frac{(b-2t)(h-2t)^3}{12} = 1.7468 \cdot 10^6 \ mm^4$$

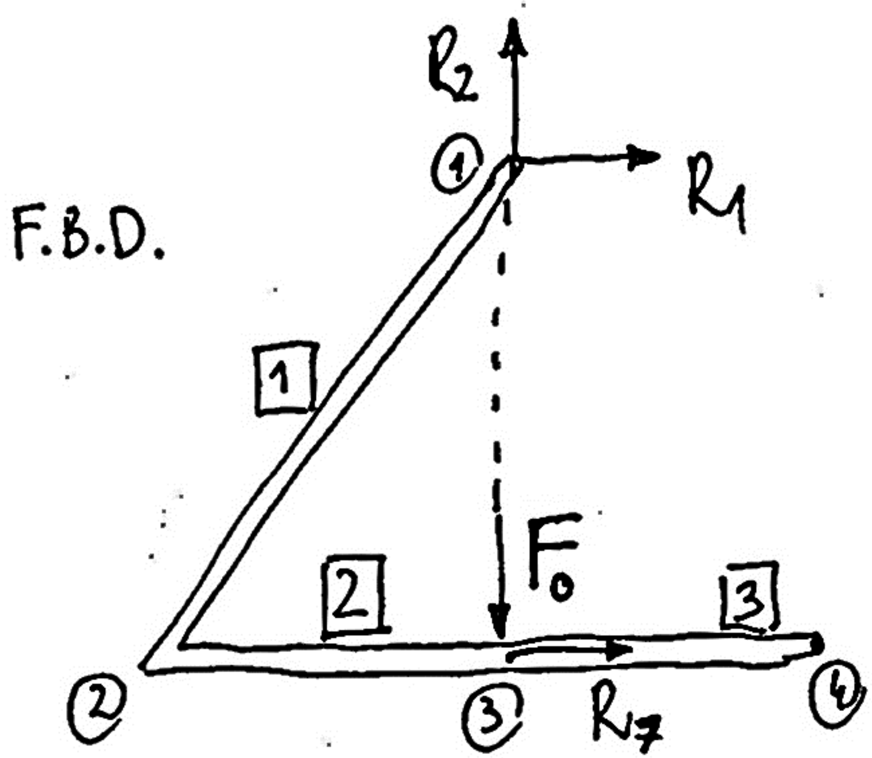
Parametry węzłowe:

$$\{q\}_{12\times 1} = \begin{Bmatrix} u_1 \\ v_1 \\ \Theta_1 \\ u_2 \\ v_2 \\ \Theta_2 \\ u_3 \\ v_3 \\ \Theta_3 \\ u_4 \\ v_4 \\ \Theta_4 \end{Bmatrix}$$

$$u_1 = 0$$
$$v_1 = 0$$
$$u_3 = 0$$

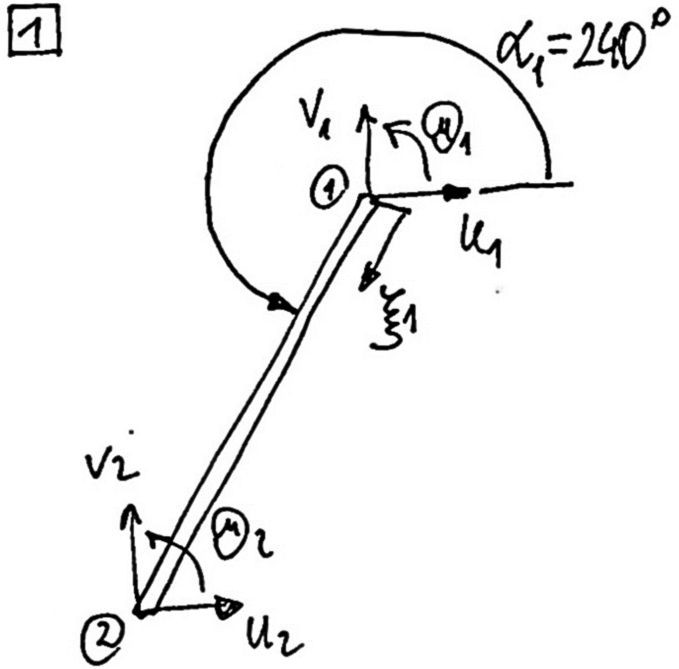


Obciążenia i reakcje:



$$\underset{12\times 1}{\{F\}} = \left\{ \begin{array}{c} R_1 \\ R_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ R_7 \\ -F_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\}$$

Macierze sztywności elementu 1:



$$C_1 = -\frac{1}{2}$$

$$S_1 = -\frac{\sqrt{3}}{2}$$

$$\underset{1\times 6}{\lfloor q_g \rfloor_1} = \lfloor u_1, v_1, \theta_1, u_2, v_2, \theta_2 \rfloor$$

$$\underset{6\times 6}{[k]_1} = \begin{bmatrix} \frac{EA}{2l} & 0 & 0 & -\frac{EA}{2l} & 0 & 0 \\ 0 & \frac{12EJ_z}{8l^3} & \frac{6EJ_z}{4l^2} & 0 & -\frac{12EJ_z}{8l^3} & \frac{6EJ_z}{4l^2} \\ 0 & \frac{6EJ_z}{4l^2} & \frac{4EJ_z}{2l} & 0 & -\frac{6EJ_z}{4l^2} & \frac{2EJ_z}{2l} \\ -\frac{EA}{2l} & 0 & 0 & \frac{EA}{2l} & 0 & 0 \\ 0 & -\frac{12EJ_z}{8l^3} & -\frac{6EJ_z}{4l^2} & 0 & \frac{12EJ_z}{8l^3} & -\frac{6EJ_z}{4l^2} \\ 0 & \frac{6EJ_z}{4l^2} & \frac{2EJ_z}{2l} & 0 & -\frac{6EJ_z}{4l^2} & \frac{4EJ_z}{2l} \end{bmatrix}$$

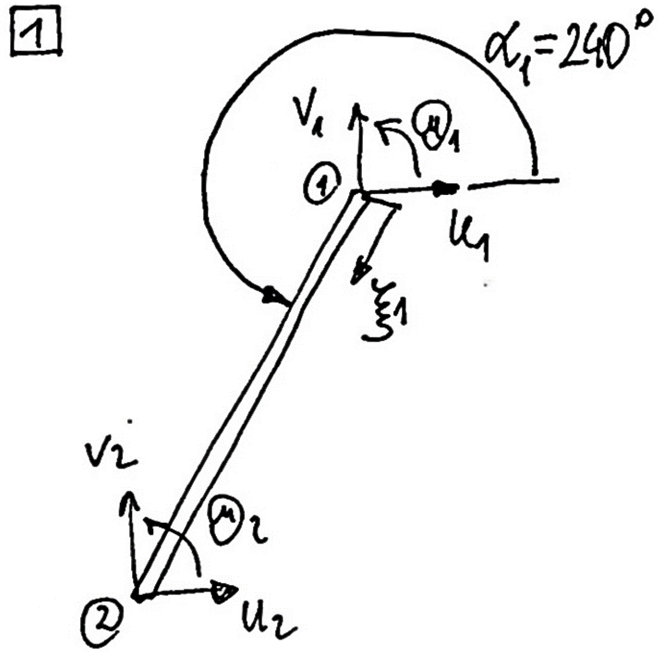
Macierz transformacji elementu 1:

$$\underset{6\times 6}{\left[T_f\right]_e} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{6\times 6}{\left[T_f\right]_1} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{6\times 6}{\left[T_f\right]_1^T} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Rozszerzona macierz sztywności elementu 1:

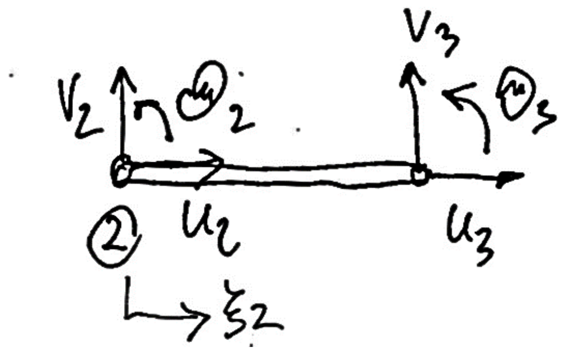


$$\underset{6\times6}{[k_g]_1} = \underset{6\times6}{[T_f]_1^T}\,\underset{6\times6}{[k]_1}\,\underset{6\times6}{[T_f]_1}$$

$$\underset{12\times12}{[k_g]_1^*} = \begin{bmatrix} \underset{6\times6}{[k_g]_1} & \underset{6\times6}{[0]} \\ \underset{6\times6}{[0]} & \underset{6\times6}{[0]} \end{bmatrix}$$

Macierz sztywności elementu 2:

$\boxed{2}$



$$\alpha_2 = 0^\circ, \quad c_2 = 1, \quad s_2 = 0$$

$$\underset{1\times 6}{\lfloor q_g \rfloor_2} = \lfloor u_2, V_2, \Theta_2, u_3, V_3, \Theta_3 \rfloor$$

$$\underset{6\times 6}{[K]_2} = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & \frac{12EJ_2}{l^3} & \frac{6EJ_2}{l^2} & 0 & -\frac{12EJ_2}{l^3} & \frac{6EJ_2}{l^2} \\ 0 & \frac{6EJ_2}{l^2} & \frac{4EJ_2}{l} & 0 & -\frac{6EJ_2}{l^2} & \frac{2EJ_2}{l} \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & -\frac{12EJ_2}{l^3} & -\frac{6EJ_2}{l^2} & 0 & \frac{12EJ_2}{l^3} & -\frac{6EJ_2}{l^2} \\ 0 & \frac{6EJ_2}{l^2} & \frac{2EJ_2}{l} & 0 & -\frac{6EJ_2}{l^2} & \frac{4EJ_2}{l} \end{bmatrix}$$

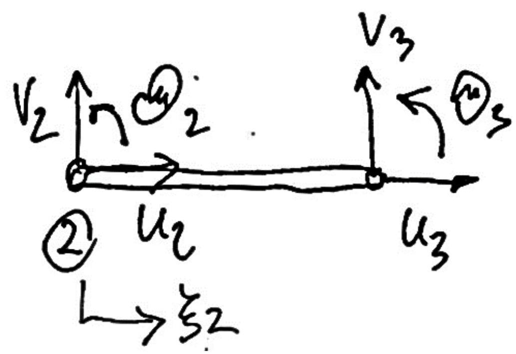
Macierz transformacji elementu 2:

$$\underset{6\times 6}{[T_f]_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \underset{6\times 6}{[T_f]_2^T}$$

$$\underset{6\times 6}{[K_g]_2} = \underset{6\times 6}{[T_f]_2^T}\,\underset{6\times 6}{[k]_2}\,\underset{6\times 6}{[T_f]_2} = \underset{6\times 6}{[k]_2}$$

Rozszerzona macierz sztywności elementu 2:

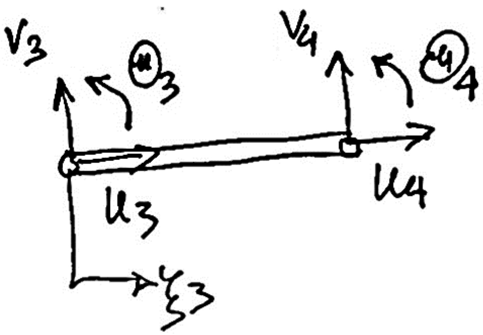
$\boxed{2}$



$$\underset{6\times 6}{\left[K_g\right]_2^{*}} = \begin{bmatrix} \underset{3\times3}{[0]} & \underset{3\times6}{[0]} & \underset{3\times3}{[0]} \\ \underset{6\times3}{[0]} & \underset{6\times6}{[K_g]_2} & \underset{6\times3}{[0]} \\ \underset{3\times3}{[0]} & \underset{3\times6}{[0]} & \underset{3\times3}{[0]} \end{bmatrix}$$

Rozszerzona macierz sztywności elementu 3:

$\boxed{3}$



$$\alpha_3 = 0^\circ, \quad c_3 = 1, \quad s_3 = 0$$

$$\underset{1\times 6}{\lfloor q_g \rfloor_3} = \lfloor u_3, v_3, \Theta_3, u_4, v_4, \Theta_4 \rfloor$$

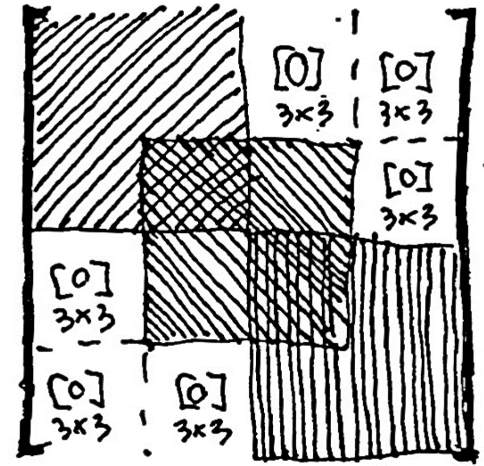
$$\underset{6\times 6}{[K]_3} = \underset{6\times 6}{[K]_2}, \quad \underset{6\times 6}{[T_f]_3} = \underset{6\times 6}{[T_f]_3^T} = \underset{6\times 6}{[T_f]_2}$$

$$\underset{6\times 6}{[K_g]_3} = \underset{6\times 6}{[K_g]_2} = \underset{6\times 6}{[K]_3}$$

$$\underset{6\times 6}{[K_g]_3^*} = \begin{bmatrix} \underset{6\times 6}{[0]} & \underset{6\times 6}{[0]} \\ \underset{6\times 6}{[0]} & \underset{6\times 6}{[K_g]_3} \end{bmatrix}$$

Globalna macierz sztywności:

$$\underset{12\times12}{[K]} = \underset{12\times12}{[K_g]_1^*} + \underset{12\times12}{[K_g]_2^*} + \underset{12\times12}{[K_g]_3^*} =$$



Układ równań:

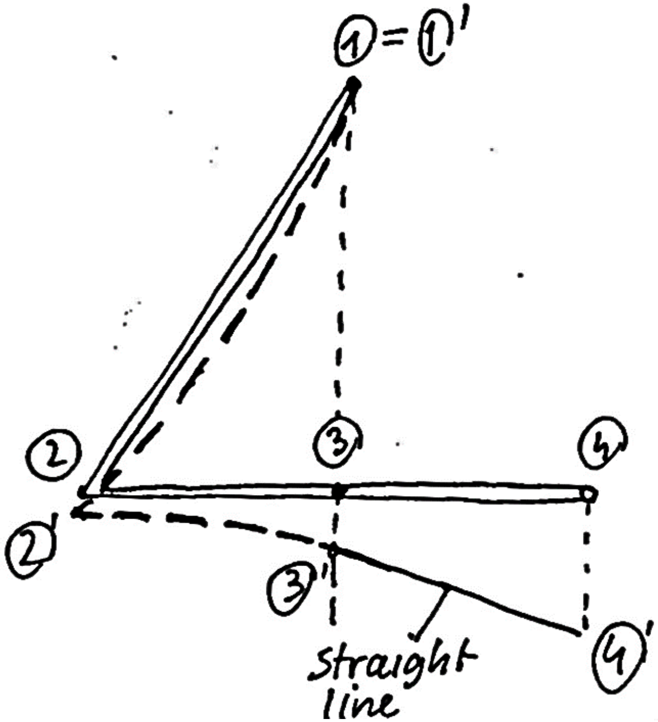
$$\underset{12\times12}{[K]} \cdot \underset{12\times1}{\{q\}} = \underset{12\times1}{\{F\}}$$ + boundary conditions

$u_1 = 0$, $v_1 = 0$, $u_3 = 0$

$$\underset{9\times9}{[K]} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \\ u_2 \\ v_3 \\ u_3 \\ u_4 \\ v_4 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -10000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\Big(\underset{9\times9}{[K]} = \underset{12\times12}{[K]}$ without rows and columns: no. 1, 2, 7 $\Big)$

Poszukiwane przemieszczenia węzłowe:



$$\underset{9\times 1}{\{q\}} = \underset{9\times 9}{[K]^{-1}} \cdot \underset{9\times 1}{\{F\}} = \begin{Bmatrix} 0.00956 \\ 0 \\ -0.0604 \\ -0.01907 \\ -28.6692 \\ -0.0334 \\ 0 \\ -62.0486 \\ -0.0334 \end{Bmatrix} \begin{matrix} \text{(rad)} \\ \text{(mm)} \\ \text{(mm)} \\ \text{(rad)} \\ \text{(mm)} \\ \text{(rad)} \\ \text{(mm)} \\ \text{(mm)} \\ \text{(rad)} \end{matrix} = \begin{Bmatrix} \varphi_1 \\ u_2 \\ v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3 \\ u_4 \\ v_4 \\ \varphi_4 \end{Bmatrix}$$

Reakcje:

$$\underset{12\times 12}{\left[K \right]} \cdot \underset{12\times 1}{\left\{ q \right\}} = \left\{ \begin{array}{c} R_1 \\ R_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ R_7 \\ -F_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\}$$

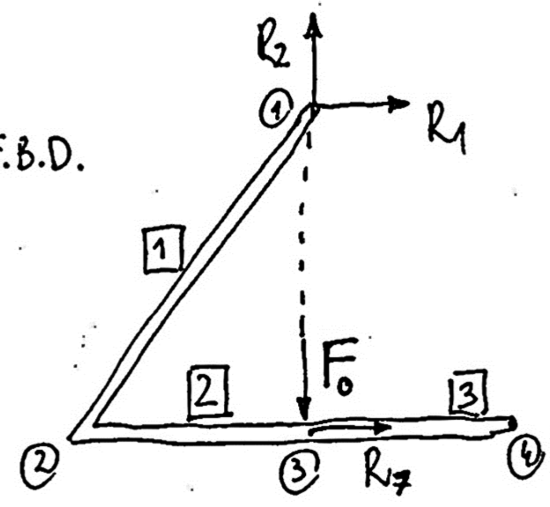
$R_1 \Rightarrow R_1 = 0\ \mathrm{N}$

$R_2 \Rightarrow R_2 = 10000\ \mathrm{N}$

$R_7 \Rightarrow R_7 = 0\ \mathrm{N}$

(additional constraint $u_3 = 0$ did not change the deformation)

F.B.D.



Przemieszczenia elementu 1 w układzie lokalnym:

$\boxed{1}$

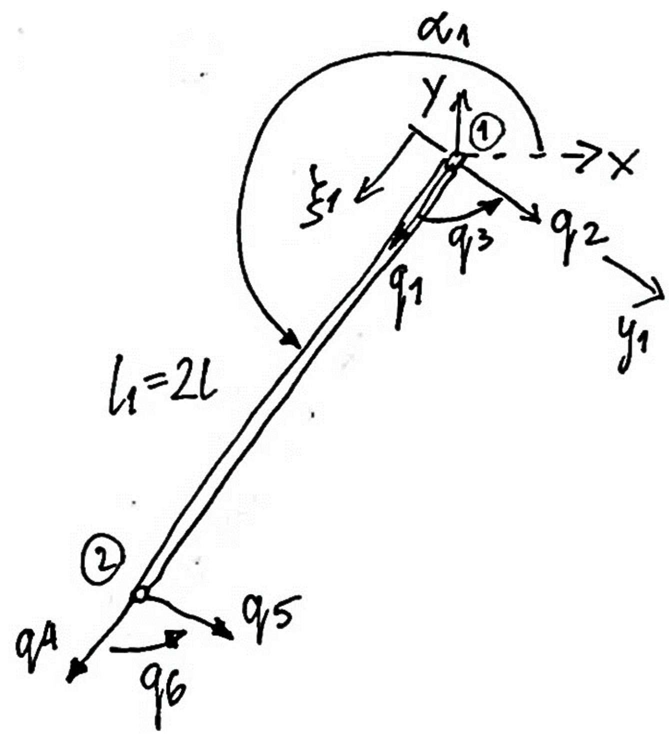
$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_1 = \underset{6\times 6}{\left[T_f\right]_1} \cdot \underset{6\times 1}{\left\{q_g\right\}_1} = \underset{6\times 6}{\left[T_f\right]_1} \cdot \begin{Bmatrix} u_1 \\ v_1 \\ \Theta_1 \\ u_2 \\ v_2 \\ \Theta_2 \end{Bmatrix}_1 =$$

$$= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0.00956 \\ 0 \\ -0.0604 \\ -0.01907 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0.00956 \\ 0.0523 \\ 0.0302 \\ -0.01907 \end{Bmatrix}_1$$

Naprężenia i siły wewnętrzne w elemencie 1:

$\boxed{1}$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0.00956 \\ 0.0523 \\ 0.0302 \\ -0.01907 \end{Bmatrix}_1$$



Pręt rozciągany:

$$\sigma_1 = \frac{E}{2l}(q_4 - q_1)_1 = 5.23 MPa$$

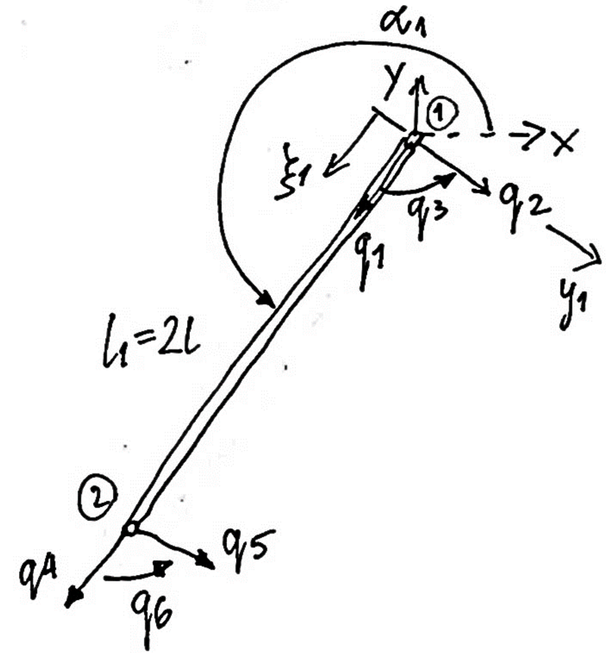
$$N_1 = \sigma_1 \cdot A = 8660.254\ N$$

Siły wewnętrzne w elemencie 1 od zginania:

$$M_{z_1}(\xi_1) = EJ_z \cdot w_1'' = EJ_z \cdot \underset{1\times 4}{\lfloor N'' \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_1 =$$

$$= EJ_z \left\lfloor -\frac{6}{l_1^2} + \frac{12}{l_1^3}\xi_1,\ -\frac{4}{l_1} + \frac{6}{l_1^2}\xi_1,\ \frac{6}{l_1^2} - \frac{12}{l_1^3}\xi_1,\ -\frac{2}{l_1} + \frac{6}{l_1^2}\xi_1 \right\rfloor \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_5 \\ q_6 \end{Bmatrix}_1 \Rightarrow$$

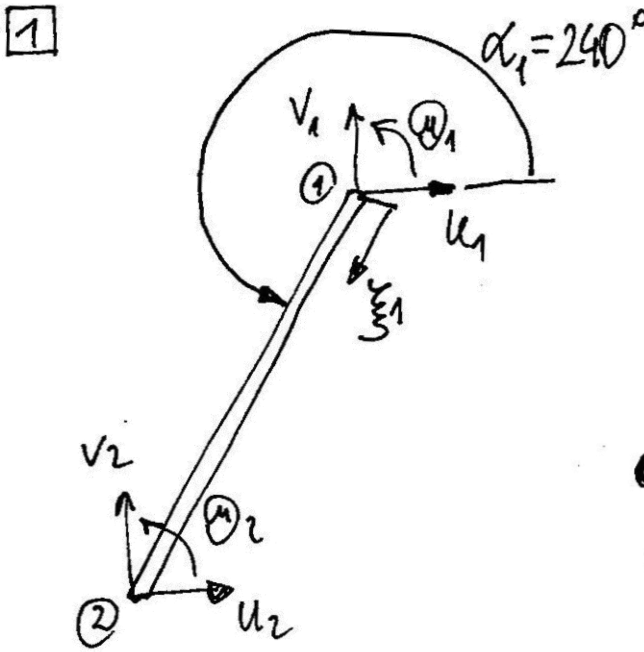
$$\Rightarrow \quad M_{z_1}(0) = 0\ , \quad M_{z_1}(2l) = -1\cdot 10^7\ \mathrm{Nmm}$$

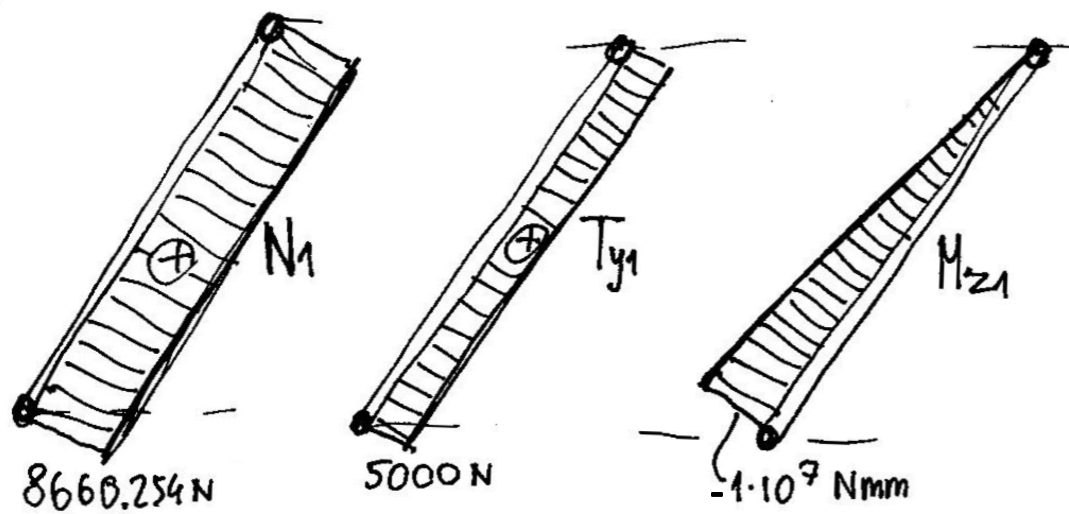


$$T_{y_1} = -EJ_z \cdot w_1''' = -EJ_z \cdot \underset{1\times 4}{\lfloor N''' \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_1 =$$

$$= -EJ_z \left\lfloor \frac{12}{l_1^3},\ \frac{6}{l_1^2},\ -\frac{12}{l_1^3},\ \frac{6}{l_1^2} \right\rfloor \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_1 = 5000\,N$$

Siły wewnętrzne i naprężenia w elemencie 1 (wypadkowe)

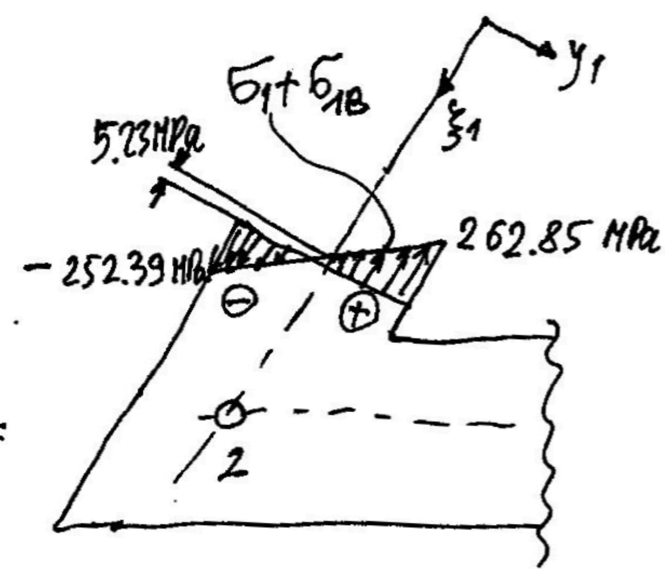




$$\sigma_{1B}(\xi_1, y_1) = -\frac{M_{z1}(\xi_1)\cdot y_1}{J_z}$$

$$\sigma_{1B}\left(2l, \frac{h}{2}\right) = \frac{-(-1\cdot 10^7\,\text{Nmm})\cdot \frac{h}{2}}{J_z} =$$

$$= 257{,}62\ \text{MPa}$$



Naprężenia i siły wewnętrzne w elemencie 2:

$\boxed{2}$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_2 = \underset{6\times 6}{[T_f]_2} \cdot \underset{6\times 1}{\{q_g\}_2} = \underset{6\times 6}{[T_f]_2} \cdot \begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{Bmatrix}_2 =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ -0.0604 \\ -0.01907 \\ 0 \\ -28.6692 \\ -0.03334 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ -0.0604 \\ -0.01907 \\ 0 \\ -28.6692 \\ -0.03334 \end{Bmatrix}_2$$

Naprężenia i siły wewnętrzne w elemencie 2:

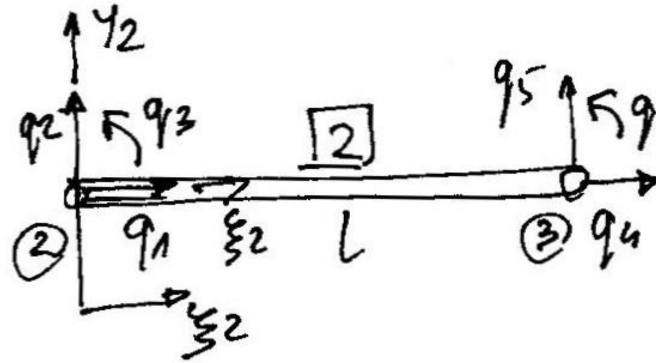
$\boxed{2}$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ -0.0604 \\ -0.01907 \\ 0 \\ -28.6692 \\ -0.03334 \end{Bmatrix}_2$$

Pręt rozciągany:

$$\sigma_2 = \frac{E}{L}(q_4 - q_1)_2 = 0$$

$$N_2 = \sigma_2 \cdot A = 0$$



Siły wewnętrzne w elemencie 2 od zginania:

$$M_{z_2}(\xi_2) = EJ_2 \cdot w_2'' = EJ_2 \cdot \underset{1\times 4}{\lfloor N'' \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_2 =$$

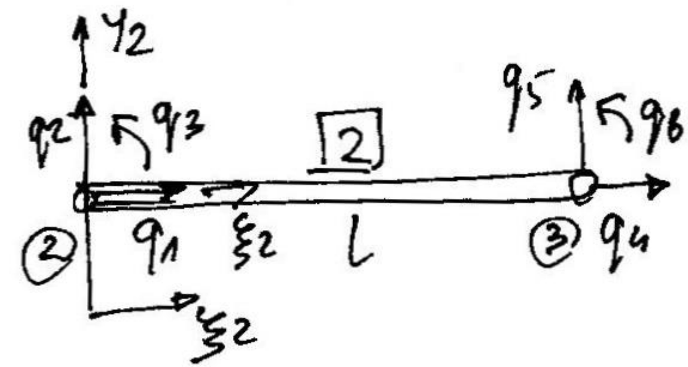
$$= EJ_2 \left\lfloor -\frac{6}{l^2} + \frac{12}{l^3}\xi_2, \; -\frac{4}{l} + \frac{6}{l^2}\xi_2, \; \frac{6}{l^2} - \frac{12}{l^3}\xi_2, \; -\frac{2}{l} + \frac{6}{l^2}\xi_2 \right\rfloor \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_2 \Rightarrow$$

$$M_{z_2}(0) = -1 \cdot 10^7 \, \mathrm{Nmm} \quad ; \quad M_{z_2}(l) = 0$$

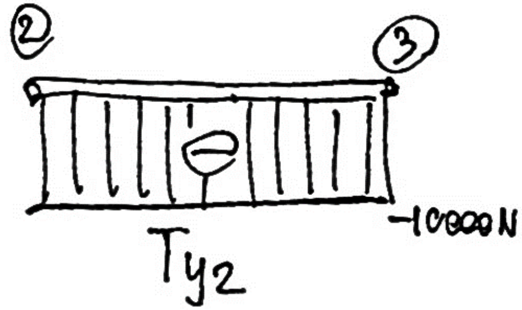
$$T_{y_2} = -EJ_2 \cdot w_2''' = -EJ_2 \lfloor N''' \rfloor \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_2 =$$

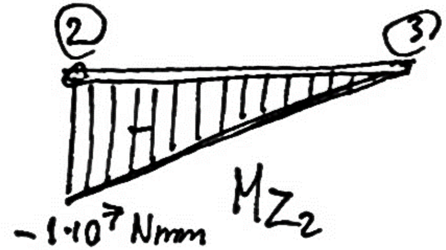
$$= -EJ_2 \left\lfloor \frac{12}{l^3}, \frac{6}{l^2}, -\frac{12}{l^3}, \frac{6}{l^2} \right\rfloor \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_2 =$$

$$= -10000 \, \mathrm{N}$$



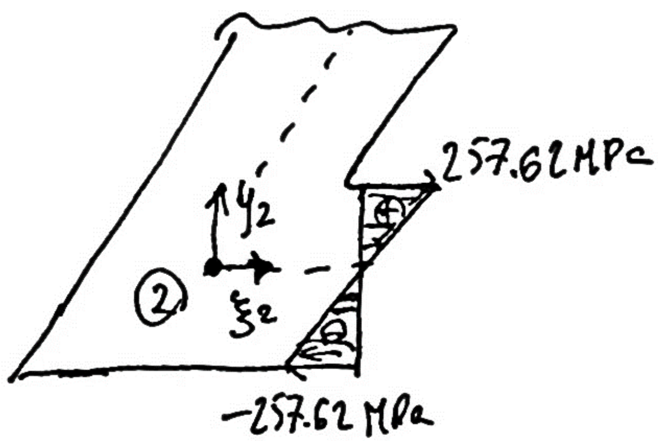
Siły wewnętrzne i naprężenia w elemencie 2 (wypadkowe)





$$\sigma_{2B}(\xi_2, y_2) = -\frac{M_{z_2}(\xi_2)\cdot y_2}{J_z}$$

$$\sigma_{2B}\left(0, \frac{h}{2}\right) = -\frac{(-1\cdot 10^{7}\,\mathrm{Nmm})\cdot\frac{h}{2}}{J_z} = 257.62\ \mathrm{MPa}$$



Naprężenia i siły wewnętrzne w elemencie 3:

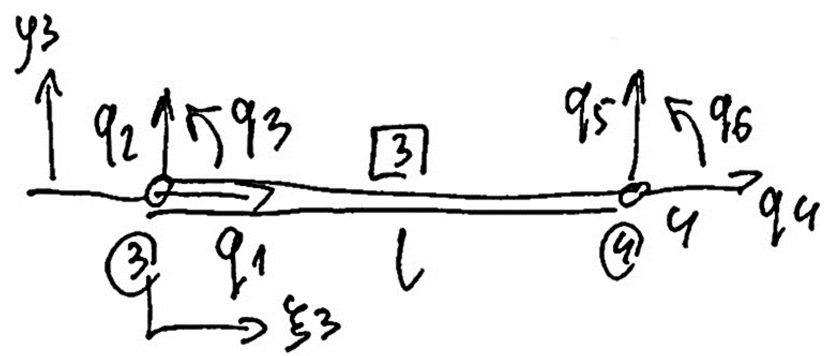
$\boxed{3}$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_3 = \underset{6\times6}{[T_f]_3} \cdot \underset{6\times1}{\{q_g\}_3} = \underset{6\times6}{[1]} \cdot \begin{Bmatrix} u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ -28.6692 \\ -0.0334 \\ 0 \\ -62.0486 \\ -0.0334 \end{Bmatrix}_3$$

Pręt rozciągany:

$$\sigma_3 = \frac{E}{l}(q_4 - q_1)_3 = 0$$

$$N_3 = \sigma_3 \cdot A = 0$$

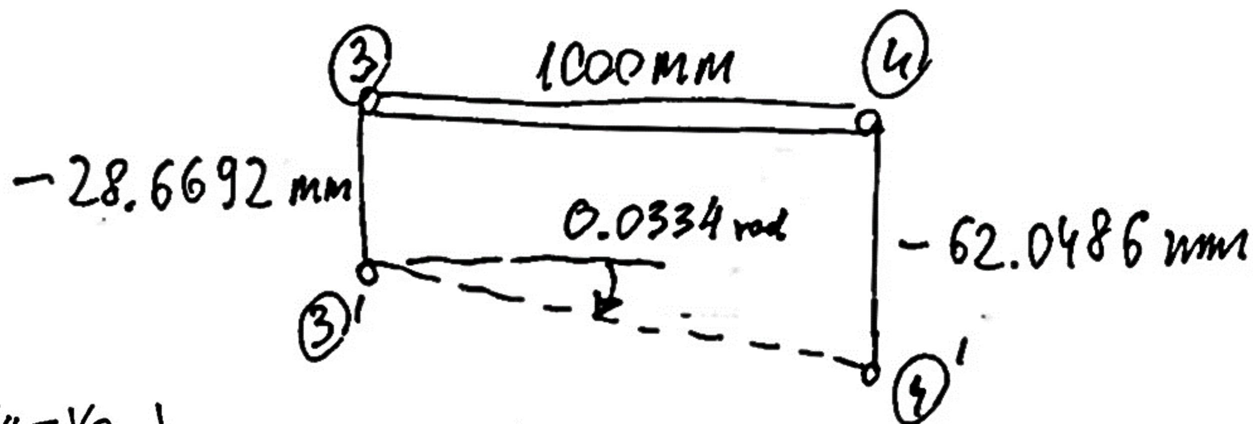


Siły wewnętrzne w elemencie 3 od zginania:

$$M_{z3}(\xi_3) = EJ_z \cdot w_3'' = EJ_z \cdot \underset{1\times 4}{\lfloor N'' \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_3 =$$

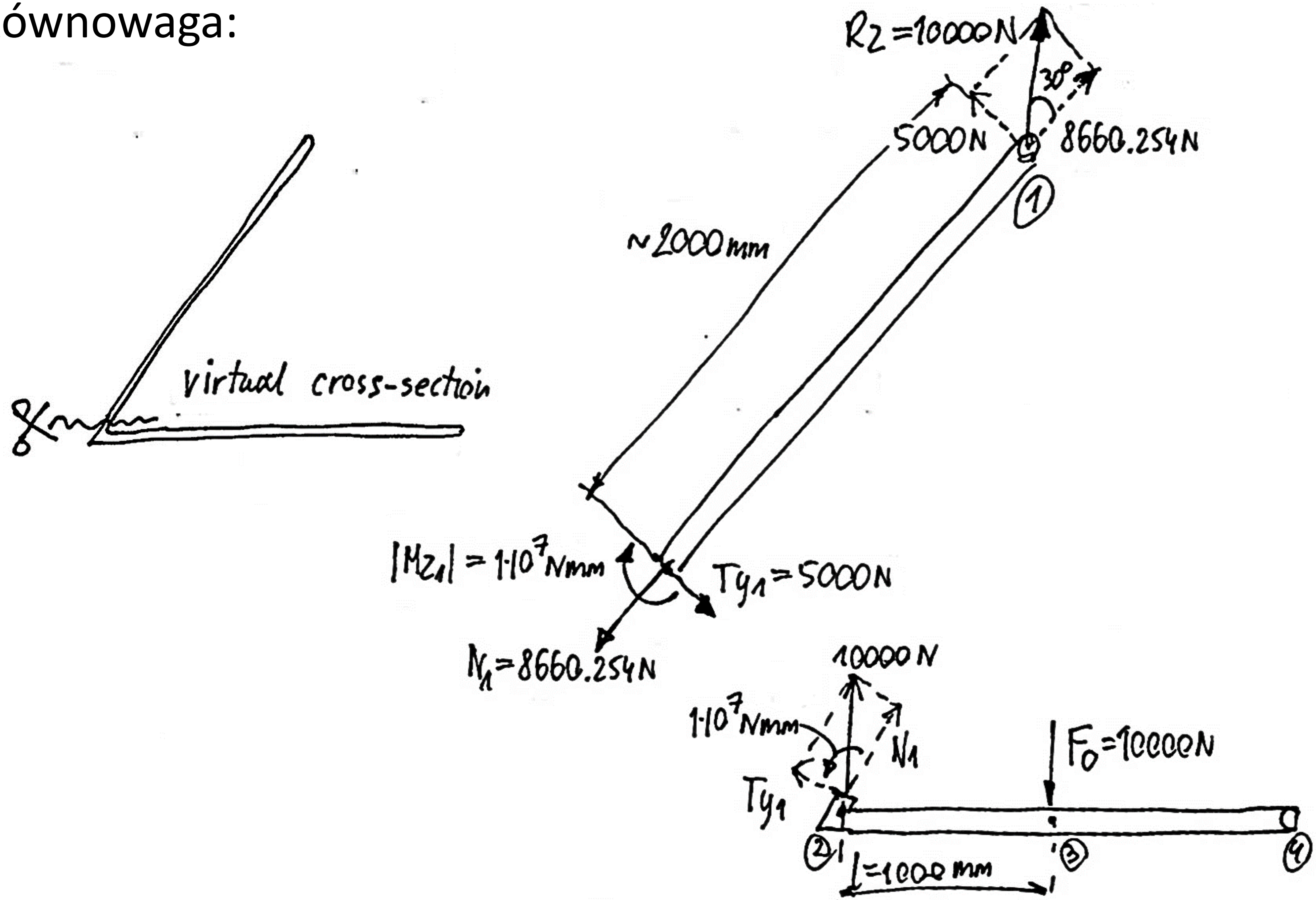
$$= EJ_z \left\lfloor -\frac{6}{l^2} + \frac{12}{l^3}\cdot\xi_3,\ -\frac{4}{l} + \frac{6}{l^2}\cdot\xi_3,\ \frac{6}{l^2} - \frac{12}{l^3}\xi_3,\ -\frac{2}{l} + \frac{6}{l^2}\xi_3 \right\rfloor \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_3 =$$

$$= 0 \quad \Rightarrow \quad \sigma_{3B}(\xi_3, y_3) = 0$$



$$\operatorname{atan}\left(\frac{v_4 - v_3}{l}\right) = -0.0334\ \text{rad}$$

Równowaga:


virtual cross-section
$R_2 = 10000N$
30°
5000N
8660.254N
1
~2000 mm
$|M_{z1}| = 1\cdot10^7 Nmm$
$T_{y1} = 5000N$
$N_1 = 8660.254N$
10000 N
$1\cdot10^7 Nmm$
N_1
T_{y1}
$F_0 = 10000N$
2
3
4
l=1000 mm